Riemannian Geometry $\boxed{\text{Week 9}}$

Last week: We proved the Hopf-Rinow theorem.

Thm (Hopf-Rinow)

Let (M,g) be connected. Then the following are equivalent:

a) (M,g) is geodesically complete.

b) $\exists p \in M$ s.t every maximal geodesic through p is complete.

c) Every closed ball in (M, d_g) is compact.

d) (M,d) is a complete metric space.

(Synthetic geometry: Using properties like geodesics, distances etc.
Hopf-Rinow is part of a "synthetic" view of geometry.)

Curvature of a Riemannian manifold:

We will study the curvature of a Riemannian manifold from an intrinsic viewpoint: Relative acceleration of nearby geodesics, angles of triangles, ...

Definition: The Riemann curvature tensor of a connection ∇:
(1,3) tensor field:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

So: $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$

Proposition: R is (indeed) a tensor field.

Proof: It suffices to prove it is C^∞-linear in all its arguments.

Recall: $[f\cdot X, Y] = f\cdot[X,Y] - Y(f)\cdot X$

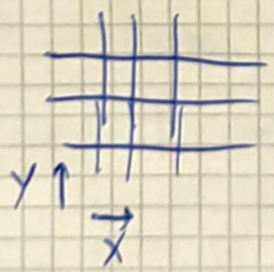
So
$$R(f\cdot X, Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX} Z - \nabla_{[fX,Y]}Z$$
$$= f\cdot\nabla_X\nabla_Y Z - \nabla_Y(f\cdot\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X}Z$$
$$= f\cdot\nabla_X\nabla_Y Z - \cancel{Y(f)\cdot\nabla_X Z} - f\nabla_Y\nabla_X Z$$
$$- f\cdot\nabla_{[X,Y]}Z + \cancel{Y(f)\cdot\nabla_X Z}$$
$$= f\cdot R(X,Y)Z.$$

Similarly for the other arguments. ∎

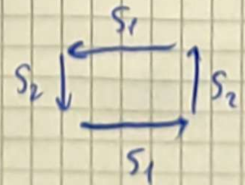
We will be only interested in the Riemann curvature tensor of the Levi-Civita connection of (M,g).

Geometric interpretation:

If X, Y are coordinate vector fields $([X,Y]=0)$:



$\gamma^{(s_1,s_2)}$: Coordinate loop



Then $\dfrac{\mathbb{P}_\gamma Z - Z}{s_1\cdot s_2} \xrightarrow{s_1,s_2\to 0} R(X,Y)\cdot Z$

(Note: Since $\mathbb{P}_\gamma: T_pM \to T_pM$ is an linear isometry, $R(X,Y): T_pM\to T_pM$ is skew-symmetric.)

Using the "musical isomorphisms":

We can write R as a $(0,4)$ tensor

(Recall: $X^a \longrightarrow \hat{X}_a = g_{a\beta} X^\beta$, $X \to g(X, \cdot)$ commutes with ∇)

Def: $\hat{R}(X,Y,Z,W) = g(R(X,Y)\cdot W, Z)$

$$\left(= -g(R(X,Y)\cdot Z, W) \right)$$

Note: • Different authors have different sign conventions

• We will often drop the $\wedge$

Coordinate notation:

• $R^a{}_{\beta\gamma\delta}$ where

$$R^a{}_{\beta\gamma\delta} = \left(R(\partial_\gamma, \partial_\delta)\cdot \partial_\beta\right)^a \quad \text{or} \quad R(\partial_\gamma, \partial_\delta)\cdot\partial_\beta = R^a{}_{\beta\gamma\delta}\cdot \partial_a$$

• $\hat{R}_{a\beta\gamma\delta} = \hat{R}(\partial_a, \partial_\beta, \partial_\gamma, \partial_\delta)$

(So for $\hat{R}$ we adopt the usual definition of the components).

As we will see: Due to the symmetries of R,

$$\hat{R}_{a\beta\gamma\delta} = R_{a\beta\gamma\delta} := g_{a\lambda} R^\lambda{}_{\beta\gamma\delta} = g(\partial_a, R(\partial_\gamma, \partial_\delta)\partial_\beta)$$

Lemma:

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\gamma\lambda}\Gamma^{\lambda}_{\beta\delta} - \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\lambda}_{\beta\gamma}$$

and $$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial^2_{\beta\gamma} g_{\alpha\delta} - \partial^2_{\alpha\gamma} g_{\beta\delta} + \partial^2_{\alpha\delta} g_{\beta\gamma} - \partial^2_{\beta\delta} g_{\alpha\gamma}\right) + \partial g \cdot \partial g$$

Proof: $$R^{\alpha}{}_{\beta\gamma\delta} = \left(R(\partial_\gamma, \partial_\delta)\partial_\beta\right)^{\alpha} =$$

$$= \left(\nabla_{\partial_\gamma}\nabla_{\partial_\delta}\partial_\beta - \nabla_{\partial_\delta}\nabla_{\partial_\gamma}\partial_\beta\right)^{\alpha}$$

$$= \partial_\gamma\left(\nabla_{\partial_\delta}\partial_\beta\right)^{\alpha} + \Gamma^{\alpha}_{\gamma\lambda}\left(\nabla_{\partial_\delta}\partial_\beta\right)^{\lambda} - (\gamma \leftrightarrow \delta)$$

$$= \partial_\gamma\Big(\partial_\delta \underbrace{(\partial_\beta)^{\alpha}}_{=\delta^{\alpha}_{\beta}} + \Gamma^{\alpha}_{\delta k}\underbrace{(\partial_\beta)^{k}}_{=\delta^{k}_{\beta}}\Big) + \Gamma^{\alpha}_{\gamma\lambda}\left(\partial_\delta(\partial_\beta)^{\lambda} + \Gamma^{\lambda}_{\delta k}(\partial_\beta)^{k}\right)$$
$$- (\gamma \leftrightarrow \delta)$$

$$= \partial_\gamma \Gamma^{\alpha}_{\delta\beta} + \Gamma^{\alpha}_{\gamma\lambda}\Gamma^{\lambda}_{\delta\beta} - (\gamma \leftrightarrow \delta) \qquad \boxtimes$$

Note: ~~The Riemann~~ $R = 0$ on $(\mathbb{R}^n, g_E)$. Conversely, (M, g) is locally isometric to $(\mathbb{R}^n, g_E)$ iff $R = 0$.

Rmk: In normal coordinates around p: $g_{ij}|_p = \delta_{ij}$, $\partial g|_p = 0$

So $$R_{\alpha\beta\gamma\delta}|_p = \frac{1}{2}\left(\partial^2_{\beta\delta} g_{\alpha\delta} - \dots\right)|_p$$

In addition: By Gauss lemma:

In normal coordinates (not just at p): $g_{ij}x^i = \delta_{ij}x^i$

$$\Rightarrow \left(\partial^2_{\alpha\beta} g_{\gamma\delta} + \partial^2_{\gamma\alpha} g_{\beta\delta} + \partial^2_{\beta\gamma} g_{\alpha\delta}\right)|_p = 0$$

So: In normal coordinates at p: $R_{\alpha\beta\gamma\delta}|_p = \left(\partial^2_{\beta\delta} g_{\alpha\gamma} - \partial^2_{\beta\delta} g_{\alpha\gamma}\right)|_p$ $\left(^2 - \langle\gamma,\gamma\rangle\right.$

Prop: Symmetries of the Riem. curvature tensor:

- $R(X,Y)\cdot Z = -R(Y,X)\cdot Z$
- $\langle R(X,Y)\cdot W, Z\rangle = -\langle R(X,Y)Z, W\rangle$
- $\hat{R}(X,Y,Z,W) = \hat{R}(Z,W,X,Y)$
- 1st Bianchi: $R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,Z,X,W) = 0$
- 2nd Bianchi: $(\nabla_X R)(Y,Z,V,W) + (\nabla_Z R)(X,Y,V,W)$
$$+ (\nabla_Y R)(Z,X,V,W) = 0$$

Proof: Suffices to prove in normal coordinates at p.

They follow directly from the explicit expression. $\square$

So: $R(X,X,Z,W) = 0$ for instance

When $\dim = 2$: Only 1 independent component.

Sectional curvature:

Prop: Let $p \in (M,g)$, $\Pi \subseteq T_pM$ be a tangent plane spanned by the non-colinear vectors $X, Y \in T_pM$.

Then the quantity
$$K(X,Y) = \frac{R(X,Y,X,Y)}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$$
is independent of the choice of $X, Y \in \Pi$

Rmk: The sectional curvature of Π is denoted with $K(\Pi)$. We also denote $\|X\wedge Y\|^2 := \|X\|^2\|Y\|^2 - \langle X,Y\rangle^2$

Proof: Let e_1, e_2 be an orth. basis of Π,

so that $X = X^1 e_1 + X^2 e_2$

$Y = Y^1 e_1 + Y^2 e_2$

Then due to the symmetries of R:

$$R(X,Y,X,Y) = \left(\det\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}\right)^2 R(e_1, e_2, e_1, e_2)$$

and $\|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2 = \left(\det\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}\right)^2$. $\square$

Ricci curvature: $Ric(X,Y) = tr(Z \longrightarrow R(Z,X) \cdot Y)$

i.e. $Ric_{ij} = R^a{}_{iaj}$. If e_i orth. basis:
$Ric(X,Y) = \sum_i R(X, e_i, Y, e_i)$

$R(X,Y) = R(Y,X)$ (see exercises)

Scalar curvature: $S = tr\, R = g^{ij} Ric_{ij}$

If constant sectional curvature K (e.g. sphere):

$Ric_{ij} = (n-1) K \cdot g_{ij}$, $S = n(n-1) \cdot K$.